Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Graph Theory

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Max. points : 40. **Time Limit : 3 hours.**

Answer any two questions only.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class or assignments, mention it clearly. See the end of the question paper for notations.

1. (a) Compute the eigenvalues of the laplacian matrix L and adjacency matrix A of the Petersen graph. Calculate the number of spanning trees of the Petersen graph. (Hint : Show that $A^2 + A - 2I = J$) (**10**)

   (b) If G has the degree sequence $d_1 \geq \ldots \geq d_n$ then show that $\chi(G) \leq 1 + \max_{1 \leq i \leq n} \min\{d_i, i-1\}$. (**10**)

2. (a) Let G be a connected graph on n vertices and m edges. Let ∂_1 be its incidence matrix under a fixed orientation of edges. Let $y = (y_1, \ldots, y_n)^T$ be a $(n \times 1)$-column vector such that for some $i \neq j \in [n]$, $y_i = +1, y_j = -1$ and $y_l = 0$ for $l \in [n] \setminus \{i, j\}$. Show that there exists a $(m \times 1)$-column vector x such that $\partial_1 x = y$. Give a graph-theoretic interpretation of the same. (**10**)

   (b) Let $p_r(G)$ denote the number of partitions of $V(G)$ into r non-empty independent sets. Show that $\chi_G(x) = \sum_{r=1}^{n} p_r(G) x_{(r)}$ where $x_{(r)} = x(x-1)\ldots(x-r+1)$ and $n = |V(G)|$. (**10**)

3. Let G be a simple, undirected graph and $s \neq t \in V(G)$.

   (a) Consider the network with all edge-capacities equal to 1. Show that if f is an integral flow from s to t of strength at least k then there exist k edge disjoint paths from s to t. (**5**).

   (b) State and prove Menger's theorem for edge disjoint paths from s to t in G. (**15**)

4. Let G be a simple, undirected graph. Given vertices s, t, a, b, we define $\tau(s, a, b, t)$ to be the number of spanning trees T of G in which $(a, b) \in T$ and the unique path from s to t goes through a and then b. Fix two vertices $s, t \in V$. Define $I(a, b) := \tau(s, a, b, t) - \tau(s, b, a, t)$ for $(a, b) \in \overrightarrow{E}$, where $\overrightarrow{E} = \{(a, b), (b, a) : a \sim b\}$ is the directed edge-set. Show that I is a flow from s to t in G and find the strength of the flow. (**20**)

Some notations :

- $[n] = \{1, \ldots, n\}$.
- G is assumed to be a finite, undirected, simple graph everywhere.
- $\chi(G)$ - chromatic number of the graph G.
- $\chi_G(.)$ - chromatic polynomial of the graph G.
- I - Identity matrix. J - Matrix with all entries 1.